MATH 5061 Lecture on 4/22/2020

Setup: (M^m, g) closed (cpt w/o boundary), oriented Riem. manifold.

de Rham complex

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "d"] & \Omega^{p-1} \arrow[l, shift left, "\delta"] \arrow[loop above, "\Delta"] \arrow[r, shift left, "d"] & \Omega^{p} \arrow[l, shift left, "\delta"] \arrow[loop above, "\Delta = d\delta + \delta d"] \arrow[r, shift left, "d"] & \Omega^{p+1} \arrow[l, shift left, "\delta"] \arrow[loop above, "\Delta"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

Note:

$\langle d\alpha, \beta \rangle_{L^2}$

$\shortparallel$

$\langle \alpha, \delta\beta \rangle_{L^2}$

$\forall \alpha \in \Omega^{p-1}$,

$\forall \beta \in \Omega^{p}$

Hodge Decomposition: $\Omega^p = \mathcal{H}_p \underset{\perp_{L^2}}{\oplus} d\Omega^{p-1} \underset{\perp_{L^2}}{\oplus} \delta\Omega^{p+1}$

$\Rightarrow$ **Hodge Thm:** $H^p_{dR}(M) \cong \mathcal{H}_p := \{ \alpha \in \Omega^p \mid \Delta\alpha = 0 \}$

Besides the Hodge Laplacian $\Delta := d\delta + \delta d$, there is another natural 2^{nd} order differential operator on Ω^p, called rough Laplacian :

$$D^2 : \Omega^p(M) \longrightarrow \Omega^p(M)$$

$$D^2\alpha := tr(DD\alpha)$$

$$= \sum_{i=1}^{m} D_{e_i} D_{e_i}\alpha - D_{D_{e_i}e_i}\alpha \qquad \{e_i\} \text{ O.N.B. for } TM$$

There is a relation between Δ & D^2 :

Bôchner - Weitzenböck formula on $\Omega^p(M)$:

$$(WF) \quad \boxed{\Delta\alpha = \overbrace{-D^2}^{D^*D}\alpha - \sum_{i,j=1}^{m} \omega^i \wedge \iota_{e_j}(R(e_i,e_j)\alpha)} \qquad \forall \alpha \in \Omega^p(M).$$

Hodge Laplacian

Rough Laplacian

p-form

where $\{e_1, \ldots, e_m\}$ $\overset{\text{local}}{\vee}$ O.N.B. for TM, dual $\{\omega^1, \ldots, \omega^m\}$ $\overset{\text{dual}}{\vee}$ O.N.B for T^*M.

ι_X : interior product w.r.t X $\quad [(\iota_X\alpha)(X_1, \ldots, X_{p-1}) := \alpha(X, X_1, \ldots, X_{p-1})]$

$R(X,Y)\alpha := D_X D_Y \alpha - D_Y D_X \alpha - D_{[X,Y]}\alpha$.

Note: $D : \Gamma(\Lambda^p T^*M) = \Omega^p(M) \longrightarrow \Gamma(T^*M \otimes \Lambda^p T^*M)$ on L^2 inner product space

$\rightsquigarrow$ "formal adjoint": $D^* : \Gamma(T^*M \otimes \Lambda^p T^*M) \longrightarrow \Omega^p(M)$

s.t. $\langle D\alpha, \gamma \rangle_{L^2} = \langle \alpha, D^*\gamma \rangle_{L^2}$.

FACT: $\boxed{D^2 = -D^*D}$ on $\Omega^p(M)$. ($\Rightarrow D^2$ self-adjoint)

We first observe a useful calculation:

Fix $\alpha, \beta \in \Omega^p$. Regard $X \mapsto \langle D_X\alpha, \beta \rangle$ as a 1-form $\omega \in \Omega^1(M)$.

$$\omega \in \Omega^1 \overset{\text{dual}}{\underset{\text{w.r.t. } g}{\longleftrightarrow}} \omega^\# \in \mathfrak{X}(M) \text{ vector field.}$$

Then, $\boxed{\operatorname{div} \omega^\# = \langle D\alpha, D\beta \rangle + \langle D^2\alpha, \beta \rangle}$ —— $(*)$

Proof of $(*)$: Let $\{e_i\}$ be O.N.B. for TM. Then write $\omega^\# = \sum_{i=1}^m \langle D_{e_i}\alpha, \beta \rangle e_i$

$$\operatorname{div} \omega^\# := \sum_{j=1}^m \langle D_{e_j}\omega^\#, e_j \rangle$$

$$= \sum_{i,j=1}^m \langle D_{e_j}\left(\langle D_{e_i}\alpha, \beta \rangle e_i\right), e_j \rangle$$

$-\langle e_i, D_{e_j}e_j \rangle$

$\parallel$ O.N.B.

$$= \sum_{i,j=1}^m \left[\underline{e_j \langle D_{e_i}\alpha, \beta \rangle \delta_{ij}} + \underline{\langle D_{e_i}\alpha, \beta \rangle \langle D_{e_j}e_i, e_j \rangle} \right]$$

$$= \underbrace{\sum_{i=1}^m \langle D_{e_i}\alpha, D_{e_i}\beta \rangle}_{\langle D\alpha, D\beta \rangle} + \sum_{i=1}^m \langle D_{e_i}D_{e_i}\alpha, \beta \rangle$$

$$- \sum_{i=1}^m \langle D_{D_{e_i}e_i}\alpha, \beta \rangle$$

$\rightarrow \langle D^2\alpha, \beta \rangle$

$(*)$ □

So, $\langle -D^*D\alpha, \beta \rangle_{L^2} \overset{\text{adjoint}}{=} -\langle D\alpha, D\beta \rangle_{L^2} = -\int_M \langle D\alpha, D\beta \rangle \, dV_g$

$$\overset{(*)}{=} -\int_M \cancel{\operatorname{div}\omega^\#} \, dV_g + \int_M \langle D^2\alpha, \beta \rangle \, dV_g = \langle D^2\alpha, \beta \rangle_{L^2}.$$

$= 0$ div thm

$\forall \alpha, \beta \in \Omega^p$.

FACT □

Corollary to (WF): For any harmonic p-form $\alpha \in \mathcal{H}_p$, we have

$$\boxed{\frac{1}{2}\Delta|\alpha|^2 = |D\alpha|^2 + F(\alpha)}$$

where $F(\alpha) = -\sum_{i,j=1}^{m} \langle \omega^i \wedge \iota_{e_j}(R(e_i,e_j)\alpha), \alpha \rangle$

Proof of Corollary: Given $\alpha \in \mathcal{H}_p$, i.e. $\Delta\alpha \equiv 0$.

$$\frac{1}{2}\Delta|\alpha|^2 = \frac{1}{2}\sum_{i=1}^{m}\left(D_{e_i}D_{e_i}|\alpha|^2 - D_{D_{e_i}e_i}|\alpha|^2\right)$$

$$= \sum_{i=1}^{m}\Big(D_{e_i}\big(\langle D_{e_i}\alpha,\alpha\rangle\big) - \langle D_{D_{e_i}e_i}\alpha,\alpha\rangle\Big)$$

$$= \underbrace{\sum_{i=1}^{m}\langle D_{e_i}\alpha, D_{e_i}\alpha\rangle}_{|D\alpha|^2} + \underbrace{\sum_{i=1}^{m}\langle D_{e_i}D_{e_i}\alpha - D_{D_{e_i}e_i}\alpha, \alpha\rangle}_{\langle D^2\alpha,\alpha\rangle} \overset{(WF)}{=} \text{R.H.S.}$$

$\square$

When $p = 1$, $\boxed{F(\alpha) = \mathrm{Ric}(\alpha^\#, \alpha^\#)}$ ✓ for $\alpha \in \Omega^1(M)$.

Verify this: Let $\alpha \in \Omega^1(M)$.

$$(D_X D_Y \alpha)(Z) = X((D_Y\alpha)(Z)) - (D_Y\alpha)(D_X Z)$$

$$= \underline{XY(\alpha(Z))} - \underline{X(\alpha(D_Y Z))} - \underline{Y(\alpha(D_X Z))} + \alpha(D_Y D_X Z)$$

$$-\ (D_Y D_X \alpha)(Z) = \underline{YX(\alpha(Z))} - \underline{Y(\alpha(D_X Z))} - \underline{X(\alpha(D_Y Z))} + \alpha(D_X D_Y Z)$$

$$-\ (D_{[X,Y]}\alpha)(Z) = \underline{[X,Y](\alpha(Z))} - \alpha(D_{[X,Y]}Z)$$

$$(R(X,Y)\alpha)(Z) = 0 - \alpha(R(X,Y)Z)$$

Therefore, $\sum_{i,j=1}^{m} \omega^i \wedge \underline{\iota_{e_j}(R(e_i,e_j)\alpha)}$

$$= \sum_{i,j=1}^{m} \omega^i \wedge (R(e_i,e_j)\alpha)(e_j)$$

$$= \sum_{i,j=1}^{m} \omega^i \wedge \left(-\alpha\left(R(e_i,e_j)e_j\right)\right)$$

$$= \sum_{i=1}^{m} \langle \alpha, \mathrm{Ric}(e_i,\cdot)\rangle \omega^i = \mathrm{Ric}(\alpha^\#,\cdot) \xrightarrow{\langle -,\alpha\rangle} \mathrm{Ric}(\alpha^\#,\alpha^\#).$$

□

Bôchner Thm: Let (M^m, g) be a closed, orientable Riem mfd.

Suppose $\mathrm{Ric}(M,g) > 0$.

THEN, the 1st Betti number $b_1 := \dim_{\mathbb{R}} H^1_{dR}(M) = 0$.

Remarks: (i) Hodge Thm $\Rightarrow$ $\mathcal{H}_1 \cong H^1_{dR}(M) = 0$

$\Rightarrow \nexists$ non-trivial harmonic 1-form.

(ii) If we only assume $\mathrm{Ric} \geq 0$, then any harmonic 1-form must be parallel (i.e. $D\alpha \equiv 0$).

Proof: By Bôchner technique.

Suppose $\alpha \in \mathcal{H}_1$. By Corollary to (WF),

$$\frac{1}{2}\Delta|\alpha|^2 = |D\alpha|^2 + \mathrm{Ric}(\alpha^\#,\alpha^\#)$$

Integrate $\Longrightarrow$ over M

$$0 \underset{\substack{\uparrow \\ \text{div thm} \\ \because M \text{ closed}}}{=} \frac{1}{2}\int_M \Delta|\alpha|^2 dV_g = \int_M \underbrace{|D\alpha|^2 dV_g}_{\geq 0} + \int_M \underbrace{\mathrm{Ric}(\alpha^\#,\alpha^\#)\, dV_g}_{\geq 0 \text{ by assumption}} \geq 0$$

□

We then proceed to the proof of (WF).

Basic Idea: Express d and δ in terms of D.

- Step 1: identities with ∂_x
- Step 2: write d, δ in terms of D
- Step 3: Express $\Delta = d\delta + \delta d$ in terms of D.

Step 1 | Lemma:

(i) $\forall\, \theta \in \Omega^1,\ \forall\, \alpha \in \Omega^p, \beta \in \Omega^{p-1}$.

" ι is adjoint to ∧ "

$$\boxed{\langle \iota_{\theta^\#}\alpha, \beta \rangle = \langle \alpha, \theta \wedge \beta \rangle} \quad \text{——} \quad (\#\#)_1$$

(ii) $\forall\, X, Y \in \mathfrak{X}(M)$,

ι , D "almost commute"

On Ω^p,
$$\boxed{D_X \circ \iota_Y - \iota_Y \circ D_X = \iota_{D_X Y}} \quad \text{——} \quad (\#\#)_2$$

Proof of Lemma: For simplicity, we prove these for 2-forms, i.e. $p = 2$.

(i) Let $\theta \in \Omega^1$, $\beta \in \Omega^1$, $\alpha = \alpha^1 \wedge \alpha^2 \in \Omega^2$.

$$\langle \iota_{\theta^\#}\alpha, \beta \rangle = \langle \alpha^1(\theta^\#)\,\alpha^2 - \alpha^2(\theta^\#)\,\alpha^1, \beta \rangle$$
$$= \langle \alpha^1, \theta \rangle \langle \alpha^2, \beta \rangle - \langle \alpha^2, \theta \rangle \langle \alpha^1, \beta \rangle .$$

$$\langle \alpha, \theta \wedge \beta \rangle = \langle \alpha^1 \wedge \alpha^2, \theta \wedge \beta \rangle = \det \begin{pmatrix} \langle \alpha^1, \theta \rangle & \langle \alpha^1, \beta \rangle \\ \langle \alpha^2, \theta \rangle & \langle \alpha^2, \beta \rangle \end{pmatrix}$$

(ii)
$$\big(D_X(\iota_Y \alpha)\big)(Z) = X\big((\iota_Y\alpha)(Z)\big) - (\iota_Y\alpha)(D_X Z)$$
$$= \cancel{X\big(\alpha(Y, Z)\big)} - \cancel{\alpha(Y, D_X Z)}$$

$$\big(\iota_Y(D_X\alpha)\big)(Z) = (D_X\alpha)(Y, Z)$$
$$= \cancel{X\big(\alpha(Y, Z)\big)} - \alpha(D_X Y, Z) - \cancel{\alpha(Y, D_X Z)}$$

$$\Rightarrow \big[(D_X \circ \iota_Y)(\alpha) - (\iota_Y \circ D_X)(\alpha)\big](Z) = \alpha(D_X Y, Z) = (\iota_{D_X Y}\alpha)(Z) .$$

———— $\square$

Step 2 | Lemma: The operators d, δ and D are related:

$$\begin{cases} d\alpha = \sum_{i=1}^{m} \omega^i \wedge D_{e_i}\alpha \\ \delta\alpha = -\sum_{i=1}^{m} i_{e_i}(D_{e_i}\alpha) \end{cases}$$

Here: $\{e_i\}$ O.N.B. for TM
$\updownarrow$ dual
$\{\omega^i\}$ O.N.B. for T^*M

for any $\alpha \in \Omega^p$.

Formula for d on Ω^p: $(p=2)$

Recall: $(p=1)$ $\quad d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

for $\omega\in\Omega^1$.

Q: What about p-forms in general?

Cartan's Formula: $\mathcal{L}_X = d\circ\iota_X + \iota_X\circ d$ on Ω^p.

By Cartan, if $\alpha\in\Omega^2$, then

$$(\mathcal{L}_X\alpha)(Y,Z) = \big(d\circ\underbrace{\iota_X(\alpha)}_{\in\Omega^1}\big)(Y,Z) + \underbrace{(\iota_X\circ d\alpha)(Y,Z)}_{d\alpha(X,Y,Z)}.$$

So,

$$\begin{aligned}
d\alpha(X,Y,Z) &= (\mathcal{L}_X\alpha)(Y,Z) - (d(\iota_X\alpha))(Y,Z)\\
&= X(\alpha(Y,Z)) - \alpha(\mathcal{L}_XY,Z) - \alpha(Y,\mathcal{L}_XZ)\\
&\quad -\big[Y((\iota_X\alpha)(Z)) - Z((\iota_X\alpha)(Y)) - (\iota_X\alpha)([Y,Z])\big]\\
&= X(\alpha(Y,Z)) - \alpha([X,Y],Z) - \alpha(Y,[X,Z])\\
&\quad - Y(\alpha(X,Z)) + Z(\alpha(X,Y)) + \alpha(X,[Y,Z])\\
&= X(\alpha(Y,Z)) - Y(\alpha(X,Z)) + Z(\alpha(X,Y))\\
&\quad - \alpha([X,Y],Z) + \alpha([X,Z],Y) - \alpha([Y,Z],X)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big(\sum_{i=1}^m \omega^i\wedge D_{e_i}\alpha\Big)(X,Y,Z) &= \sum_{i=1}^m\Big[\omega^i(X)(D_{e_i}\alpha)(Y,Z) - \omega^i(Y)(D_{e_i}\alpha)(X,Z)\\
&\qquad + \omega^i(Z)\,(D_{e_i}\alpha)(X,Y)\Big]\\
&= \underline{(D_X\alpha)(Y,Z)} - \underline{(D_Y\alpha)(X,Z)} + \underline{(D_Z\alpha)(X,Y)}\\
&= \underline{X(\alpha(Y,Z)) - \alpha(D_XY,Z) - \alpha(Y,D_XZ)}\\
&\quad \underline{- Y(\alpha(X,Z)) + \alpha(D_YX,Z) + \alpha(X,D_YZ)}
\end{aligned}$$

$$+Z(\alpha(X,Y)) - \alpha(D_Z X, Y) - \alpha(X, D_Z Y)$$

This proves the formula for d on Ω^2.

Formula of δ on Ω^p $(p=2)$.

By def^{n}, $\langle \alpha, d\beta \rangle_{L^2} = \langle \delta\alpha, \beta \rangle_{L^2}$ $\forall\ \alpha \in \Omega^2,\ \beta \in \Omega^1$.

$$\int_M \langle \delta\alpha, \beta \rangle \, dV_g = \int_M \langle \alpha, d\beta \rangle \, dV_g \overset{\text{formula of } d}{=} \int_M \Big\langle \alpha, \sum_{i=1}^m \omega^i \wedge D_{e_i}\beta \Big\rangle \, dV_g$$

$$\overset{\text{step 1}}{=} \sum_{i=1}^m \Big[\int_M \langle \iota_{e_i}\alpha, D_{e_i}\beta \rangle \, dV_g \Big]$$

$$= \underbrace{\sum_{i=1}^m \int_M e_i \langle \iota_{e_i}\alpha, \beta \rangle \, dV_g}_{(\mathrm{I})} - \underbrace{\sum_{i=1}^m \int_M \langle D_{e_i}(\iota_{e_i}\alpha), \beta \rangle \, dV_g}_{(\mathrm{II})}$$

(I)

$$\sum_{i=1}^m e_i \langle \iota_{e_i}\alpha, \beta \rangle = \sum_{i=1}^m \Big\langle D_{e_i} \overbrace{\Big(\sum_{j=1}^m \langle \iota_{e_j}\alpha, \beta \rangle e_j \Big)}^{X}, e_i \Big\rangle$$

$$- \sum_{i,j=1}^m \langle \iota_{e_j}\alpha, \beta \rangle \underbrace{\langle D_{e_i} e_j, e_i \rangle}_{= -\langle e_j, D_{e_i} e_i \rangle}$$

$$= \operatorname{div} X + \sum_{i=1}^m \langle \iota_{D_{e_i} e_i}\alpha, \beta \rangle$$

(II)

$$\sum_{i=1}^m \langle D_{e_i}(\iota_{e_i}\alpha), \beta \rangle \overset{\text{Step 1}}{=} \sum_{i=1}^m \langle \iota_{e_i}(D_{e_i}\alpha), \beta \rangle + \sum_{i=1}^m \langle \iota_{D_{e_i} e_i}\alpha, \beta \rangle$$

swap

$$\text{So,} \quad = \int_M \cancelto{0}{\operatorname{div} X} \, dV_g - \int_M \sum_{i=1}^m \langle \iota_{e_i}(D_{e_i}\alpha), \beta \rangle \, dV_g$$

div thm

□

(WF)

$\blacksquare$

Step 3: Exercises.

Q: What about Bôchner formula on p-forms, $p \geq 2$?

Recall: Curvature Operator

$$\mathcal{R} : \Lambda^2 TM \longrightarrow \Lambda^2 TM \qquad \text{"self-adjoint"}$$

$$\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l \rangle := R(e_i, e_j, e_k, e_l).$$

$\rightsquigarrow$ quadratic form : $Q(\alpha) := \langle \mathcal{R}(\alpha), \alpha \rangle$

Def$^{\underline{n}}$: (M, g) has positive curvature operator $(\mathcal{R} > 0)$

iff $Q(\alpha) \geq 0 \quad \forall \alpha \in \Gamma(\Lambda^2 TM)$.

and "$=$" $\Leftrightarrow$ $\alpha \equiv 0$.

Note:

$$\underset{(\text{e.g. } S^m/\Gamma)}{\mathcal{R} > 0} \underset{\not\Leftarrow}{\Longrightarrow} \underset{(\text{e.g. } \mathbb{CP}^m)}{K > 0} \underset{\not\Leftarrow}{\Longrightarrow} \text{Ric} > 0 \underset{\not\Leftarrow}{\Longrightarrow} \text{Scal} > 0$$

Thm: (M^m, g) closed orientable, $\mathcal{R} > 0$

$\Rightarrow$ $b_p := \dim H^p_{dR}(M) = 0$ for $p = 1, \ldots, m-1$. $\left(\Rightarrow \tilde{M} \overset{\text{homology}}{=} S^m\right)$

"Sketch of Proof": For $\overset{}{\underset{\text{harmonic}}{\wedge}}$ p-forms, $F(\phi) = \langle \tilde{R}(\rho \mathcal{R} \rho^*) \phi, \phi \rangle \underset{\substack{\uparrow \\ \text{when } \mathcal{R} > 0.}}{\geq} 0$.

$\blacksquare$

Remark: In fact, $\tilde{M}^m \overset{\text{diffeo}}{\cong} S^m$ (Ricci flow: Hamilton '86, Böhm-Wilking '06)